Lecture 11

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1 Proof of the Triangle Removal Lemma

Proof. Consider the graph G which is ϵ-far from being triangle-free.
We start with an equipartition A of G with $\frac{5}{\epsilon}$ sets.

Set $\epsilon' = \min\{\frac{\epsilon}{5}, \gamma^{\Delta}(\frac{\epsilon}{5})\} = \frac{\epsilon}{10}$. (The $\gamma^{\Delta}(\cdot)$ function is defined in KS Lemma.)

Apply Regularity Lemma by setting the parameters as follows, $m = \frac{5}{\epsilon}$, $\epsilon = \epsilon'$ (this ϵ represents the parameter ϵ in Regularity Lemma), we have $T = T(\frac{5}{\epsilon}, \epsilon')$. By Regularity Lemma, A can be refined into an equipartition B, $B = \{V_1, V_2, ..., V_k\}$, such that,

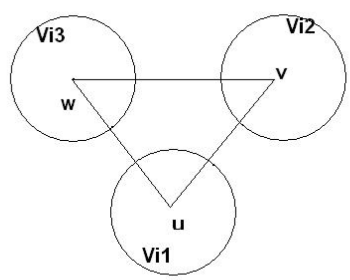
1. $\frac{5}{\epsilon} \leq k \leq T$
2. at most $\epsilon' \binom{k}{2}$ pairs of $\{V_1, V_2, ..., V_k\}$ are not ϵ'-regular
3. $|V_i| = \frac{n}{k} \in [\frac{n}{T}, \frac{\epsilon n}{5}]$, for $i = 1, 2, ..., k$

Definition 1 (useful edge)**.** *An edge (u, v), where $u \in V_i$, $v \in V_j$, is useful if it satisfies the following three conditions: (1) $i \neq j$. (2) (V_i, V_j) is ϵ'-regular. (3) The density $d(V_i, V_j) \geq \frac{\epsilon}{5}$.*

We show that G does not contain many non-useful edges.

Claim 2. *G has at most $\epsilon\binom{n}{2}$ non-useful edges with respect to equipartition B.*

We complete the proof of the Lemma by applying this claim. We remove all the non-useful edges in G. Since G is ϵ-far from being triangle-free, there are triangles left in G. Let (u, v, w) be a triangle, where $u \in V_{i_1}$, $v \in V_{i_2}$, $w \in V_{i_3}$.



By KS Lemma, the number of triangles in G is at least $\delta^{\Delta}(\frac{\epsilon}{5})|V_{i_1}||V_{i_2}||V_{i_3}|$, which is at least $\frac{\epsilon}{10}(\frac{n}{T})^3 = \delta \cdot \binom{n}{3}$.

We are left to prove the Claim. We prove the Claim by counting the number of edges which violate the three conditions respectively.

1. The number of edges n_1 violating the first condition. Each vertex in G can have at most $\frac{n}{k} - 1$ neighbors in the same partition.

$$n_1 \leq (\frac{n}{k} - 1) \cdot n \leq \frac{(n-1)n}{k} = \frac{2}{k} \cdot \binom{n}{2} \leq \frac{2\epsilon}{5} \cdot \binom{n}{2}$$

.

2. The number of edges n_2 violating the second condition. By Regularity Lemma, there are at most $\epsilon' \binom{k}{2}$ pairs of partitions in B which are not ϵ'-regular. Each of the pairs contributing at most $(\frac{n}{k})$ cross edges.

$$n_2 \leq \epsilon' \binom{k}{2} \cdot (\frac{n}{k})^2 \leq \epsilon' \cdot \frac{k(k-1)}{2} \cdot \frac{n(n-1)}{k(k-1)} = \epsilon' \binom{n}{2} \leq \frac{\epsilon}{5} \cdot \binom{n}{2}$$

.

3. The number of edges n_3 violating the third condition. For any pair V_i, V_j, $V_i \neq V_j$ and $d(V_i, V_j) < \frac{\epsilon}{5}$, $e(V_i, V_j) = d(V_i, V_j)|V_i||V_j|$,

$$n_3 \leq \binom{k}{2} e(V_i, V_j) < \frac{\epsilon}{5} \cdot (\frac{n}{k})^2 \cdot \binom{k}{2} \leq \frac{\epsilon}{5} \cdot \binom{n}{2}$$

.

Hence, $n_1 + n_2 + n_3 \leq \frac{4\epsilon}{5}\binom{n}{2} < \epsilon\binom{n}{2}$. We thus prove the Claim.

□

2 Testing other graph properties

Theorem 3 (Alon). *Let H be a fixed graph on h nodes. Let P_H be the property that a graph does not contain a copy of H as a subgraph.*
(1) If H is bipartite,

- *There is a two-sided error ϵ-tester for P_H with $O(\frac{1}{\epsilon})$ queries.*
- *There is a one-sided error ϵ-tester for P_H with $O(h^2(\frac{1}{2\epsilon})^{h^2/4})$ queries.*

(2) If H is not bipartite, then there exists $c > 0$, such that every one-sided error ϵ-tester for P_H makes $\Omega((\frac{c}{\epsilon})^{c \log \frac{c}{\epsilon}})$ queries.

Exercise 11.1 (Canonical Tester). *We define the Canonical Tester as follows,*

Algorithm 1: Canonical Tester

1. Pick s nodes uniformly and independently at random.
2. Query edges between every pair of the chosen nodes.
3. Accept or reject only based on this information.

Show that if there is an ϵ-tester T with query complexity $q(\epsilon, n)$, then there is a Canonical Tester T' with query complexity $q^2(\epsilon, n)$. Moreover, if T has one-sided error, so does T'.