Sublinear Algorithms	Feburary 23, 2012

Lecture 9

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1 In class problem solving: adaptivity in the dense-graph model

We consider graph properties which are invariant under graph isomorphism. Namely, properties $\mathcal{P}$ of graphs such that it is satisfied for G if and only if $\mathcal{P}$ is satisfied for G' obtained by permuting the vertices of the graph. Observe that this also implies the following: G is ϵ-far from $\mathcal{P}$ if and only if G' obtained by permuting the vertices of G is ϵ-far from $\mathcal{P}$. We record this in the following observation.

Observation 1 (Graph property invariant under renaming of vertices.)**.** *Consider a graph $G = (V, E)$ and let π be an arbitrary permutation of the vertex set V. Let $G' = (V, E')$ be the graph obtained by permuting vertices of G according to π. That is, $(\pi(u), \pi(v))$ is an edge in G' if and only if (u, v) is an edge in G. Then every* graph property *$\mathcal{P}$ (by definition) satisfies the following:*

- *G satisfies $\mathcal{P}$ if and only if G' satisfies $\mathcal{P}$.*
- *For every $\epsilon \in (0, 1]$, G is ϵ-far from $\mathcal{P}$ if and only if G' is ϵ-far from $\mathcal{P}$.*

We want to show that adaptivity doesn't help much in testing graph properties in the dense graph model. Recall,*nonadaptive* algorithms first decide which queries they make and only then get the input whereas *adaptive* algorithms can make queries based on answers obtained to previous queries. Goldreich and Trevisan [GT03] showed that the following tester (called the canonical tester) can be used to test any graph property (in dense graph model).

1. Pick a subset S of nodes.
2. Query all pairs (i, j) where $i, j \in S$.
3. Accept or Reject based on answers to quereis in (2).

Observe that the query complexity of the canonical tester is $O(|S|^2)$. Clearly, the tester is nonadaptive. Here we don't show the complete reduction to the canonical tester but explain how to convert an adaptive testing algorithm to non-adaptive testing algorithm. This is one of the first steps in getting a canonical tester. We state the claim formally below.

Theorem 2 (Goldreich Trevisan, [GT03])**.** *Consider any algorithm T for a* graph property *$\mathcal{P}$ that makes $q(n, \epsilon)$ queries. Then, there is a canonical tester for $\mathcal{P}$ that makes at most $2q(n, \epsilon)^2$ queries. Moreover, the transformation preserves 1-sidedness of the tester.*

Proof sketch of the reduction from adaptive to nonadaptive testers. We first give a tester T' which "simulates" T. For every entry (i, j) queried by T, if vertex i was not involved in previous queries, query (i, x) for all x involved in previous queries. Do the same for vertex j. Finally, make the same decision as T did.

Since T' simulates T (and makes the same decision as T does), it follows that T' is a tester for $\mathcal{P}$ and has the same guarantees (such as the error probabilities and in particular the guarantee of being 1-sided error tester) as T'. Let q be the number of queries made by T. Then the query complexity of T' is $\binom{2q}{2} \leq 2q^2$. (Also the running time is not significantly higher either.)

The algorithm T' is still adaptive. To get rid of adaptivity, we consider the following variation of T'. Consider a tester T'' which runs T' on a randomly permuted graph G. More formally, let π be a permutation on the set of vertices of G. Let $\pi(G)$ be the graph on the same vertex set as G such that $\pi(G)$ has an edge $(\pi(u), \pi(v))$ if and only if (u, v) is an edge in G. The algorithm T'' runs T' on $\pi(G)$ where π is a uniformly random permutation from the set of all permutations on the vertex set of G. To show that the resulting tester is nonadaptive, it helps to think of the permutation π as being chosen on the fly. One can then show that each query of T'' is uniformly distributed on the set of vertices and is independent of the queries made thus far. □

The role of adaptivity in Bounded degree graphs model Most algorithms in this model (i) either do BFS from randomly chosen nodes or, (ii) do random walks from randomly chosen nodes. It seems that these models use some weak form of adaptivity. Importantly, the following can be shown (see [RS06]): for every *interesting* graph property $\mathcal{P}$ in the bounded degree model, every nonadaptive algorithm for testing $\mathcal{P}$ must make $\Omega(\sqrt{n})$ queries.

References

[GT03] Oded Goldreich and Luca Trevisan. Three theorems regarding testing graph properties. *Random Struct. Algorithms*, 23(1):23–57, 2003.

[RS06] Sofya Raskhodnikova and Adam Smith. A note on adaptivity in testing properties of bounded degree graphs. *Electronic Colloquium on Computational Complexity (ECCC)*, 13(089), 2006.